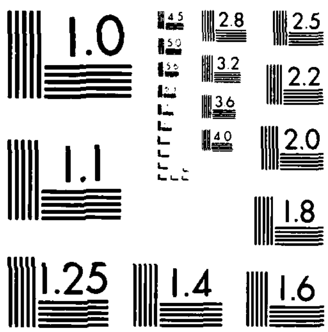
AD-A149 222 BORN INVERSION WITH A STRATIFIED REFERENCE VELOCITY(U) 1/1
COLORADO SCHOOL OF MINES GOLDEN CENTER FOR WAVE
PHENOMENA J K COHEN ET AL. 08 DEC 84 CWP-008
UNCLASSIFIED N00014-84-K-0049 F/G 12/1 NL

END
FILMED
DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS 1963 A

(12)

CSM

AD-A149 222

BORN INVERSION WITH A

STRATIFIED REFERENCE VELOCITY

by

Jack K. Cohen and Frank G. Hagin

Partially supported by the Consortium Project of the Center for Wave Phenomena and by the Selected Research Opportunities Program of the Office of Naval Research

Colorado School of Mines
Golden, Colorado 80401

Center for Wave Phenomena
Department of Mathematics
303/273-3557

12

CWP-008

BORN INVERSION WITH A

STRATIFIED REFERENCE VELOCITY

by

Jack K. Cohen and Frank G. Hagin

Partially supported by the Consortium Project of the Center for Wave Phenomena and by the Selected Research Opportunities Program of the Office of Naval Research

Center for Wave Phenomena
Department of Mathematics
Colorado School of Mines
Golden, Colorado 80401
(303) 273-3557

CONTENTS

GLOSSARY

A	inversion amplitude (10)
B	inversion amplitude ratio (28)
c_1	velocity below reflector (35)
$c(z)$	reference velocity (1)
$D, E, F, G, H(K,z)$	various integrals; see Appendix A
f	frequency (53)
$g(\underline{x},z;\underline{\xi},\omega)$	Green's function for c(z) medium (2)
$h(x)$	cylindrical surface (29)
K	ray parameter (3)
$k_3(K,z)$	$\sqrt{n^2(z)-K^2}$ (6)
$n(z) = c(0)/c(z)$,	the index of refraction (7)
R	reflection coefficient (34)
R_j	reflection coefficient (26)
S	abbreviation; see (33)
$u_S(\underline{\xi};\omega)$	observed scattered field (2)
$U_S(\xi;t)$	observed scattered field (55)
$v(\underline{x},z)$	velocity (1)
W	the inversion operator (10)
$x = (x,y)$	horizontal cartesian (1)
z	vertical cartesian (1)
$\alpha(\underline{x},z)$	unknown perturbation in velocity (1)
β	reflectivity function (26)
γ	abbreviation; see (32)
γ_1	abbreviation; see (34)
$\delta_B(s_j)$	bandlimited delta function (26)

Δc	jump in c (35)
$\underline{\xi} = (\xi,\eta)$	cartesians for observation point at $z = 0$ (2)
$\rho = \lvert\underline{x}-\underline{\xi}\rvert$	offset (4)
$\rho' = \lvert\underline{x}'-\underline{\xi}\rvert$	offset (11)
$\tau(K,z)$	travel time (8)
$\underline{\Phi}$	phase function (14)
ω	circular frequency (2)

INTRODUCTION

Carter and Frazer [1984], and Bleistein and Gray [1984] (henceforth BG), present inversion algorithms which include the effect of a stratified reference velocity, c(z). Those papers did not address the question of obtaining accurate values of the reflection coefficient; this is the issue treated here. Thus, in the language of Bleistein, Cohen and Hagin [1984], (henceforth BCH), the earlier algorithms provided structural inversions, the location of the sub-surface layers; whereas the present algorithm also provides an accurate estimate of the reflectivity function, which depicts the reflectors and provides an estimate of the reflection strengths across the layers.

Since we employ a perturbation assumption (the "Born Approximation"), the constant reference speed inversion first described in Bleistein and Cohen [1979a] and reviewed in BCH, is often not adequate at depth. Although recursive use of the algorithm is possible and although the results can be significantly enhanced by suitable pre- or post-processing (e.g., see Hagin and Cohen [1984]), extension of the perturbation method to a stratified reference profile is highly significant. It is far more likely that the actual velocity function can be well approximated by a stratified reference velocity than by a constant one, which in turn enhances the validity of the perturbation assumption and the inversion results. See BG for further discussion of this point.

The algorithm presented here has the same structure as the BG algorithm and hence it can be expected to exhibit the same stability and economy. In particular, we note that the processing times for this algorithm with depth-

dependent background velocity will be comparable to those for a constant background k-f algorithm. In addition, we shall show below that the algorithm can be expected to be quite robust even when the "small" perturbation assumption is violated.

A key feature of our approach to this problem is repeated application of high frequency asymptotic methods to obtain an inversion formula valid in the high frequency regime. Discussion of the motivation and justification for such high frequency approximation may be found in BG and BCH. In particular, we shall use a ray theoretic Green's function in formulating our basic integral equation; equation (2) below. A similar approach was presented in Clayton and Stolt [1981]. Furthermore, we cannot determine an exact inversion of the fundamental integral equation of this method. Instead, we assume an inversion operator which consists of multiplication of the observed data by a factor of the form A $\exp\{-2i\omega\tau\}$ and integration over the data set. The phase τ is the traveltime in the $c(z)$ background medium between the source/receiver point and the output point at depth. We must still determine the amplitude, A, in this operator. To do so, we require that the operator applied to Kirchhoff data from a single reflector produce the <u>reflectivity function</u> for a single reflector, to leading order, asymptotically. This leads to the determination of the amplitude of the inversion integral operator. It is to this extent that we then claim to have an inversion operator which correctly estimates reflection strength.

HIGH FREQUENCY STRUCTURAL INVERSION

Here we describe the formalism for determination of an asymptotic inversion operator, up to the amplitude, A, introduced above. We employ the same wave equation model as described in detail in BCH. If v is the velocity in the wave equation, we set

$$\frac{1}{v^2(\underline{x},z)} = \frac{1}{c^2(z)}\left[1 + \alpha(\underline{x}, z)\right] \quad , \qquad \underline{x} = (x,y) \quad . \tag{1}$$

Here c(z) is the <u>known</u>, stratified reference velocity, while $\alpha(\underline{x},z)$ is the desired perturbation correction to the actual velocity. Furthermore, we retain the assumption of <u>backscatter</u> ("stacked") data. In this case the basic integral equation for $\alpha(\underline{x},z)$ is (cf. BCH, equation (8)):

$$u_S(\underline{\xi};\omega) = \omega^2 \int\int d^2x \int_0^\infty dz \, \frac{\alpha(\underline{x},z)}{c^2(z)} \, g^2(\underline{x},z;\underline{\xi};\omega) \quad , \qquad \underline{\xi} = (\xi,\eta) \tag{2}$$

where all unmarked integral signs are over $(-\infty,\infty)$. Here u_S denotes the backscattered field at the location $\underline{\xi} = (\xi,\eta)$ on the observation plane, z = 0 and g (the "incident field") denotes the Green's function corresponding to the stratification, c(z). In contrast to the constant background case, g cannot be determined exactly: we must use the high frequency assumption. Fortunately, this assumption is completely justified on the geophysical exploration scale and has long been used to simplify processing formulas even when it was possible to derive wide band analytic results (see BCH). We use J. B. Keller's [1978] ray method formalism (see also Bleistein [1984]), which is the multi-dimensional analogue of the WKB method to obtain a parametric representation of g (see Appendix B):

$$g \sim \frac{e^{i\omega\tau(K,z)}}{4\pi\sqrt{k_3(K,0)\,k_3(K,z)\,E(K,z)\,H(K,z)}} \tag{3}$$

Here, if we introduce the transverse distance,

$$\rho = |\underline{x} - \underline{\xi}| \quad , \tag{4}$$

then the parameter, K, in (3) is defined as a function of ρ and z by the <u>ray equation</u>:

$$\rho = K\,H(K,z) \tag{5}$$

Further, the quantity $k_3(K,z)$ is given by

$$k_3(K,z) = \sqrt{n^2(z) - K^2} \quad , \quad (K^2 < n^2(z)) \tag{6}$$

where in turn, n, the index of refraction, is

$$n(z) = \frac{c(0)}{c(z)} \quad . \tag{7}$$

The travel-time, τ, is

$$\tau = \frac{1}{c(0)}\,G(K,z) \quad , \qquad G = \int_0^z \frac{n^2(\zeta)\,d\zeta}{k_3(K,\zeta)} \quad , \tag{8}$$

and finally, the quantities E and H are likewise integrals involving n and k_3. These integrals, as well as others that occur subsequently, are defined in Appendix A. There, we also derive some needed relations involving these quantities.

We shall not need the extension of k_3 in (6) to the range $n^2(z) > K^2$ because our Fourier transforms are integrals over real wave numbers only.

Thus our task is to invert the integral equation,

$$u_S(\xi,\omega) = \frac{\omega^2}{16\pi^2} \int\int d^2x \int_0^\infty dz \, \frac{\alpha(\underline{x},z)}{c^2(z)} \frac{e^{2i\omega\tau(K,z)}}{k_3(K,0)\, k_3(K,z)\, E(K,z)\, H(K,z)} \tag{9}$$

for $\alpha(\underline{x},z)$ in terms of the data, $u_S(\xi,\omega)$. Again, the ray parameter $K = K(\rho,z)$ is defined by (5).

Since the phase in (9) resembles that of a Fourier transform, we are motivated to seek an asymptotic inversion operator of the form:

$$W[f(\xi,\omega)](\underline{x}',z') \sim \int\int d^2\xi \int d\omega \, A(\rho',z') \, e^{-2i\omega\tau(K',z')} \, f(\xi,\omega) \quad , \tag{10}$$

with ρ' defined as

$$\rho' = |\underline{x}' - \xi| \quad , \tag{11}$$

and K' defined by

$$\rho' = K'E(K',z') \quad . \tag{12}$$

Here we have introduced primes to avoid confusion with the integration variables in (9). Applying W to both sides of (9) and writing out the right hand explicitly we have:

$$W[U_S](\underline{x}',z') \sim \frac{1}{16\pi^2}\int_0^\infty dz \int d\omega \iint d^2x \iint d^2\xi$$

$$\frac{\alpha(\underline{x},z)}{c^2(z)}\,\omega^2\,A(\rho',z')\,\frac{\exp\{2i\omega\Phi(K,z,K',z')/c(0)\}}{k_3(K,0)\,k_3(K,z)\,E(K,z)\,H(K,z)}\quad, \tag{13}$$

where we have introduced the phase,

$$\Phi = G(K,z) - G(K',z')\quad . \tag{14}$$

And K, K' are defined by (5) and (12) respectively.

We perform four dimensional stationary phase in $\underline{x} = (x,y)$ and $\underline{\xi} = (\xi,\eta)$ recognizing that K depends on $\underline{x}$ and $\underline{\xi}$ while K' depend/ $\underline{\xi}$. We present some of the details here to give the reader the flavor o^ the type of mathematical analysis involved. Those so inclined may skip directly to the result given by equation (21) below. Noting that (5) and (12) allow computation of $\partial K/\partial x$, $\partial K/\partial y$, $\partial K/\partial z$, etc., by implicit differentiation, and using the result of Appendix A, we find

$$\Phi_x = G_K\,\frac{\partial K}{\partial x} = G_K\,\frac{x-\xi}{\rho(KE)_K} = KH\,\frac{x-\xi}{\rho H} = \frac{x-\xi}{E}\quad . \tag{15}$$

Similarly,

$$\Phi_y = \frac{y-\eta}{E},\qquad \Phi_\xi = \frac{\xi-x}{E} - \frac{\xi-x'}{E'},\qquad \Phi_\eta = \frac{\eta-y}{E} - \frac{\eta-y'}{E'} \tag{16}$$

where we have introduced the short-hand

$$E = E(K,z),\quad E' = E(K',z)\quad . \tag{17}$$

Thus the stationarity conditions are

$$\underline{x} = \underline{\xi} = \underline{x}' \quad . \tag{18}$$

But this and (5) imply (see discussion below following equation 48):

$$K = 0 \quad . \tag{19}$$

Similiarly,

$$K' = 0 \quad . \tag{20}$$

These results greatly simplify the remainder of the stationary phase calculation (see Appendix D for the details) which yields,

$$\begin{aligned} W[u_S](\underline{x}',z') \sim \frac{1}{16} \int_0^\infty dz \; \alpha(\underline{x}',z) \; A(0,z') \; n(z) \; \frac{E(0,z')}{E(0,z)} \\ \cdot \int d\omega \; \exp\{2i \frac{\omega}{c(0)} \int_{z'}^{z} n(\zeta) d\zeta\} \quad . \end{aligned} \tag{21}$$

It should be realized that since we have already employed the assumption of high frequency several times in obtaining (21) and shall use it again before we finish, the ω integration must be construed as a band-limited integration. These matters have been discussed in Cohen and Bleistein [1979b], Bleistein [1984], and BCH [1984]. For the present, we shall merely symbolize the effect of band-limiting by placing a subscript B on the Dirac delta function which results from the ω-integration in (21). Thus,

$$\begin{aligned} \int d\omega \; \exp\{2i(\omega/c(0)) \int_{z'}^{z} n(\zeta) d\zeta\} &= \pi c(0) \; \delta_B\left(\int_{z'}^{z} n(\zeta) d\zeta\right) \\ &= \pi c(0) \; \frac{\delta_B(z-z')}{n(z)} \end{aligned} \tag{22}$$

and so (21) becomes

$$W[u_S](\underline{x}',z') \sim \frac{\pi c(0)}{16} A(0,z')\alpha(\underline{x}',z') \quad . \tag{23}$$

Solving for α, writing out the expression for $W[u_S]$(cf. equation 10), and dropping the now superfluous primes yields:

$$\alpha(\underline{x},z) \sim \frac{16}{\pi c(0)} \int\int d^2\xi \int d\omega \frac{A(\rho,z)}{A(0,z)} e^{-2i\omega\tau(K,z)} u_S(\underline{\xi},\omega) \tag{24}$$

with K determined by

$$\rho = K E(K,z); \quad \rho \equiv |\underline{x} - \underline{\xi}| \quad . \tag{25}$$

As discussed in BCH, once we surrender knowledge of the low frequency input information, we cannot obtain output trend information. It is to be hoped that (by iteration if necessary) our c(z) reference velocity is an adequate approximation of the trend to the depths of interest. What we <u>can</u> obtain from band-limited information is a perturbation correction consistent with the model of jumps across a series of interfaces. We determine the approximate location of these interfaces as well as the approximate value of the reflection coefficient at the interfaces. This information is summed up in the reflectivity function,

$$\beta = \sum R_j \delta_B(s_j) \tag{26}$$

where s_j is a (local) arclength variable measured normally from the j^{th} interface and R_j is the normal reflection coefficient of that interface. Clearly, knowledge of β is equivalent to knowledge of reflector location and the normal reflection coefficient (see equation (43) below). In turn the latter allows direct computation of the jump in c across the reflector.

According to the theory developed in Cohen and Bleistein (1979b) and reviewed in BCH, we can obtain β from α by inserting a factor of $i\omega/2c(z)$ in (24) to obtain:

$$\beta(\underline{x},z) \sim \frac{8i}{\pi c(0)c(z)} \int\int d^2\xi\ B(\rho,z) \int d\omega\ \omega\ e^{-2i\omega\tau(K,z)}\ u_S(\underline{\xi},\omega) \tag{27}$$

with K determined by (25). Here, since the inversion depends only on the ratio of A's in (24), we have introduced

$$B(\rho,z) = \frac{A(\rho,z)}{A(0,z)} \quad . \tag{28}$$

However, and this is the key point: at this stage of the derivation we have no information about $B(\rho,z)$! Any choice of this quantity gives, in the language of BCH, a <u>structural inversion</u>, i.e., a migration. In order to determine a choice of B which will yield an accurate approximation of the interface reflection coefficients, we will insert in (27), a <u>canonical</u> set of scattering data, u_S, and then determine B by enforcing the asymptotic equality in (27). For this purpose we will use Kirchhoff data for a single reflecting surface.

DETERMINATION OF THE INVERSION AMPLITUDE

In order to find an amplitude $B(\rho,z)$ for the inversion formula (27), we first obtain an expression for the Kirchhoff representation of data U_s. Such data employs the high frequency assumption (by using the multi-dimensional WKB representation of the incident, reflected and transmitted fields), but does not make the Born approximation of small reflection coefficient. Thus, if we can determine a B which enforces asymptotic equality in (27) for such data, our algorithm is likely to be quite robust for large contrast interfaces.

It remains to decide on the surfaces to use in computing the Kirchhoff approximation to u_S. It turns out that a single surface, such as the tilted plane, $z = z_0 - x \tan \beta$, would suffice to determine B. However, we carry out our calculations for the more general cylindrical (i.e. y independent) surface:

$$z = h(x) \quad . \tag{29}$$

One may think of the tilt-plane members (which are included in (29)) as determining B, while the remaining members confirm this choice of B.

In Appendix C, we show that the back scatter at ξ from (29) has the Kirchhoff (high frequency) representation:

$$u_S(\xi,\omega) \sim 2i\omega \iint d^2\bar{x} \sqrt{1+h'^2(\bar{x})}\; \gamma R S\, e^{2i\omega\tau(\bar{K},\bar{x})} \tag{30}$$

subject to

$$\bar{p} = \bar{K}E(\bar{K},\bar{z}) \quad , \qquad \bar{p} \equiv |\underline{\bar{x}} - \underline{\xi}|, \ \bar{z} \equiv h(\bar{x}) \quad . \tag{31}$$

Here we have used bars to distinguish the spatial variable from the output variables in the inversion formula (27) and have also introduced the quantities:

$$\gamma = \frac{\left[\dfrac{(\bar{x}-\xi)h'}{E(\bar{K},\bar{z})} - k_3(\bar{K},\bar{z}) \right]}{\left[c(0)\sqrt{1+h'^2} \right]} \quad , \tag{32}$$

$$S = \frac{1}{16\pi^2 k_3(\bar{K},0)k_3(\bar{K},\bar{z})E(\bar{K},\bar{z})H(\bar{K},\bar{z})} \quad , \tag{33}$$

and the (non-normal) reflection coefficient,

$$R = \frac{\gamma-\gamma_1}{\gamma+\gamma_1} \quad , \quad \gamma_1 = \mathrm{sgn}(\gamma)\sqrt{\gamma^2 + \frac{1}{c_1^2} - \frac{1}{c^2}} \quad . \tag{34}$$

In turn, $c_1(z)$ denotes the actual velocity below the reflector, $\bar{z} = h(\bar{x})$, that is,

$$c_1(z) = c(z) + \Delta c \tag{35}$$

where Δc is <u>not</u> accounted for by the stratified reference profile $c(z)$. Obviously determining R is tantamount to determining Δc or $c_1(z)$. Combining (27) and (28), we have

$$\beta(\underline{x},z) \sim \frac{-16}{\pi c(0)c(z)} \int d\omega\ \omega^3 \int\int d^2\xi \int\int d^2\bar{x}$$

$$\sqrt{1+h'^2}\ \gamma R S B\ e^{2i\omega\left[\tau(\bar{K},\bar{z})\ -\ \tau(K,z)\right]} \tag{36}$$

Our goal is to determine B so as to obtain the asymptotic equality.

We now carry out stationary phase in $\bar{x}$, $\bar{y}$, ξ, η (see Appendix D for details) and obtain the stationarity conditions:

$$\bar{y} = \eta = y \quad , \tag{37}$$

$$\bar{x} - \xi = -\mathrm{sgn}(h')\bar{K}\ E(\bar{K},\bar{z}) \quad , \tag{38}$$

and

$$x - \xi = -\ \mathrm{sgn}(h')\ K E\,(K,z) \quad . \tag{39}$$

These imply:

$$K = \bar{K} = \frac{n(\bar{z})\,|h'|}{\sqrt{1+h'^2}} \quad , \tag{40}$$

$$k_3(\bar{K},\bar{z}) = \frac{n(z)}{\sqrt{1+h'^2}} \quad , \tag{41}$$

where again $\bar{z} = h(\bar{x})$.

Geometrically, these conditions confirm the cylindrical nature of our reflector and show that the output point $(\underline{x},z)$ lies on a <u>specular</u> ray. Furthermore (37-39) yield

$$\gamma = -\frac{1}{c} \quad , \quad \gamma_1 = -\frac{1}{c_1} \tag{42}$$

which, in turn, imply that R reduces to the normal reflection coefficient,

$$R = R_n = \frac{c_1 - c}{c_1 + c} \tag{43}$$

Completing the stationary phase analysis (again see Appendix D) we find that

$$\beta(\underline{x},z) \sim \frac{16\pi n^2(z)\sqrt{1+h'^2}\, R_n S B(\rho,z)}{c(0)\sqrt{\left|\det\Phi_{ij}\right|}} \cdot \int d\omega\, e^{2i\omega[\tau(\bar{K},\bar{z}) - \tau(\bar{K},z)]} \tag{44}$$

Here,

$$\det\Phi_{ij} = \frac{1}{E(\bar{K},z)E(\bar{K},\bar{z})}\left[\frac{(1+h'^2)^2}{H(\bar{K},z)H(\bar{K},\bar{z})} + \frac{n(\bar{z})h''}{\sqrt{1+h'^2}}\left[\frac{1}{H(\bar{K},z)} - \frac{1}{H(\bar{K},\bar{z})}\right]\right] \tag{45}$$

However, the final integration in ω, yields a delta function whose argument can be transformed to arclength along rays:

$$\begin{aligned}\int d\omega\, e^{2i\omega[\tau(\bar{K},\bar{z}) - \tau(\bar{K},z)]} &= \pi\delta_B[\tau(\bar{K},\bar{z}) - \tau(\bar{K},z)] \\ &= \pi c(0)\ \delta_B[G(\bar{K},\bar{z}) - G(\bar{K},z)] \\ &= \pi c(0)\ \frac{k_3(\bar{K},z)}{n^2(z)}\ \delta_B[z - \bar{z}] \\ &= \pi c(0) \cdot \frac{1}{n(z)}\, \delta_B[s(z) - s(\bar{z})] \quad .\end{aligned} \tag{46}$$

Here the last equality involving the ray arclength variable follows from

8) and Appendix B. We may note that since $z = \bar{z}$ when the delta function "acts," the stationarity condition (38-39) imply that the output point $(\underline{x}, z)$ coincides with the specular point $(\bar{\underline{x}}, \bar{z}) = (\bar{\underline{x}}, h(\bar{x}))$. Furthermore, when $z = \bar{z}$ the second term in the square brackets in (45) drops out and hence using the definition (33), we see that

$$\frac{S}{\sqrt{|\det \Phi_{ij}|}} = \frac{1}{16\pi^2(1+h'^2)k_s(\bar{K},0)k_s(\bar{K},\bar{z})} \quad . \tag{47}$$

Hence (44) reduces to

$$\begin{aligned} \beta(\underline{x},z) &\sim \frac{R_n n(\bar{z})\ B_s}{\sqrt{1+h'^2}\ k_s\ (\bar{K},0)\ k_s(\bar{K},\bar{z})} \delta[s(z) - s(\bar{z})] \\ &= \frac{R_n B_s}{k_s(\bar{K},0)} \delta[s(z)-s(\bar{z})] \qquad (48) \\ &= \frac{R_n B_s}{\sqrt{1-\bar{K}^2}} \delta[s(z) - s(z)] \quad . \end{aligned}$$

Here, the middle equality follows from (41), while the final inequality follows from the definitions (6-7) when $z = 0$. Also the quantity B_s denotes the value of B at the stationary point and when $z = h(x)$. Thus to enforce asymptotic equality in our inversion algorithm (27), we need to choose a B which reduces to $\sqrt{1-\bar{K}^2}$ under these conditions: Obviously such a B is

$$B(\rho,z) = \sqrt{1-K^2} \tag{49}$$

where, as usual, the parameter K is determined by (25). Even though the algorithm (27) depends only on B, and not on the $A(\rho,z)$ originally

introduced in (10), self-consistency demands that we be able to write B in the form (28). This is indeed simple to do since, for example, if we pick

$$A(\rho,z) = \sqrt{1-K^2} \tag{50}$$

Then by (25), and (A-7) and (A-15), the unique K corresponding to $\rho = 0$ is $K = 0$. Thus,

$$A(0,z) = 1 \tag{51}$$

and (49) verifies (28). The fact that A is non-unique is irrelevant since (27) only depends on the ratio B.

It should be noted that choosing B as in (49) only guarantees that (27) gives an accurate estimate of β at the specular point (stationary point). While this is asymptotically the most important point, any practical implementation of (27) involves integrating over a <u>region</u> including the specular point. It is conceivable that a still more accurate B could be discovered by somehow determining it without use of the canonical Kirchhoff data. We have made a considerable effort to find a suitable generalization of Fourier inversion which would allow a derivation along the lines originally presented in Cohen and Bleistein (1979a) and reviewed in BCH. Unfortunately, to date, these efforts have not succeeded.

REMARKS ON DATA PROCESSING

The algorithm for the reflectivity function derived in the previous sections is

$$\beta(\underline{x},z) \sim \frac{8i}{\pi c(0)\ c(z)} \int\int d^2\xi \ \sqrt{1-K^2} \int d\omega\ \omega\ e^{-2i\omega\tau(K,z)}$$

$$\cdot \int_0^\infty dt\ U_S(t,\xi)\ e^{i\omega t} \quad , \tag{52}$$

$$\rho = KE(K,z) \quad , \quad \rho \equiv |\underline{x} - \underline{\xi}| \ .$$

Here the integrals E and G are defined in Appendix A and U_S is the backscatter data observed on an areal array. For actual data processing, it is convenient to "fold" the unphysical negative frequencies onto the positive ones by replacing ω by $-\omega$ on the interval $(-\infty,0)$. At the same time, we introduce the physical frequency variable (measured in Hz):

$$f = \frac{\omega}{2\pi} \tag{53}$$

and explicitly acknowledge the bandlimiting by introducing F(f), a tapered high pass filter. After these changes, we have:

$$\beta(\underline{x},z) \sim \frac{-64\pi}{c(0)c(z)} \int\int d^2\xi\ \sqrt{1 - K^2}$$

$$\cdot \ \mathrm{Im} \int_0^\infty df\ f\ F(f)\ e^{-4\pi i f\tau(K,z)}$$

$$\cdot \int_0^\infty dt\ U_S(t,\underline{\xi})\ e^{2\tau i f t} \quad , \tag{54}$$

$$\rho = KE(k,z) \quad , \quad \rho \equiv |\underline{x} - \underline{\xi}| \ .$$

In practice, areal observations are often not available and instead only a linear set of data is used. In this case we cannot hope to reconstruct a three dimensional image of the subsurface and instead seek a two dimensional slice, $\beta(x,0,z) \equiv \beta(x,z)$, consistent with the data available. Since the data is now independent of η,

$$U_S(t,\underline{\xi}) = U_S(t,\xi,0) \equiv U_S(t,\xi) \quad , \tag{55}$$

and we may carry out an additional stationary phase calculation in η. The stationarity condition is

$$\eta = y \tag{56}$$

and the analogue of (52) is found to be (see Appendix D for details):

$$\begin{aligned} \beta(x,z) \sim \frac{8}{\sqrt{\pi c(0)}\; c(z)} \int d\xi \sqrt{(1-K^2)E(K,z)} \quad \cdot \\ \int d\omega \sqrt{i\omega}\; e^{-2i\omega\tau(K,z)} \\ \int_0^\infty dt\; U_S(t,\xi)\; e^{i\omega t} \quad , \end{aligned} \tag{57}$$

$$\rho = K E(K,z) \quad , \quad \rho \equiv |x-\xi| \quad ,$$

while (54) becomes:

$$\beta(x,z) \sim \frac{32\pi}{\sqrt{c(0)}\ c(z)} \int d\xi \sqrt{(1-K^2)\ E(K,z)} \quad .$$

$$(\mathrm{Re} - \mathrm{Im}) \int_0^\infty df \sqrt{f}\ F(f)\ e^{-4\pi i f \tau(K,z)}$$

$$\cdot \int_0^\infty dt\ U_S(t,\xi)\ e^{2\pi i f t} \quad , \tag{58}$$

$$\rho = K E(K,z) \quad , \quad \rho \equiv |x-\xi| \quad .$$

The basic concepts of reducing (58) to a computer code are the same as those discussed in BG for the algorithm presented there. Briefly, the t and f integrals are performed routinely using an efficient FFT algorithm. The main complication in (58) lies in the expressions $E(K,z)$ and $\tau(K,z)$, both being integrals defined by (A-4) and (8) respectively. This is a bit subtle in that the parameter K (see Appendix B) can be viewed as determining the starting angle for a *ray connecting* the surface point $(\xi,0)$ to data point (x,z) in (58). Therefore, for a given offset $\rho = |x-\xi|$, K is defined by the implicit relation $\rho = K E(K,z)$. In the computation this issue is handled quite efficiently by two tables for evaluating $\tau(K,z)$ and the amplitude (involving E) in (58) as functions of ρ and z.

The computation time of the resulting algorithm, as pointed out in BG, is comparable to a standard k-f migration algorithm for constant coefficients.

We are now developing and testing a very carefully designed Fortran 77 computer code for (58) and expect to present the results of those tests in the near future.

CONCLUSIONS

We have presented the derivation of an inversion algorithm for backscattered ("stacked") seismic data. We made four major assumptions: (i) the acoustic wave equation is an adequate model, (ii) backscattered data has amplitude information worth preserving fairly accurately, (iii) the actual reflectivity coefficients can be adequately modeled as perturbations from a continuous reference velocity which depends only on the depth variable, (iv) the subsurface can be adequately modeled as a series of layers with jump discontinuities in the velocity (or impedance) at these layers.

The last assumption is unavoidable given the nature of the high pass (on the exploration scale) data collected in the field. The third assumption is inherent in our approach although, as pointed out above, the algorithm can be expected to be robust even when this assumption is violated. Also the algorithm presented here represents a considerable improvement over earlier algorithms, such as Cohen and Bleistein [1979a], which perturbed from a <u>constant</u> reference velocity.

On the other hand, weakening of the first two assumptions seems eminently feasible and we hope to apply the techniques expanded in this article to both inversion of offset data ("inversion before stack") and to equations which more accurately describe the wave propagation in the earth.

It is already known (see BG) that algorithms with the structure of the one presented here are numerically stable and are computationally efficient

relative to other seismic data processing algorithms.

The essence of the derivation presented is to form an inversion operator with a suitable pre-specified phase, but unknown amplitude. The amplitude is then determined by demanding that for high frequency synthetic scattering data from a fairly general surface, the algorithm asymptotically produce the correct reflectivity.

ACKNOWLEDGMENT

The authors gratefully acknowledge the support of the Consortium Project on Seismic Inverse Methods For Complex Structures at the Center for Wave Phenomena, Colorado School of Mines. Consortium members are Amoco Production Company, Conoco, Inc., Geophysical Exploration Company of Norway A/S, Golden Geophysical Corp., Marathon Oil Company, Mobil Research and Development Corp., Phillips Petroleum Company, Sun Exploration and Research, Texaco USA, Union Oil Company of California, Western Geophysical, and the Selected Research Opportunities Program of the Mathematical Branch of the Office of Naval Research.

REFERENCES

Bleistein, N., Handelsman, R.A., 1975, Asymptotic expansion of integrals: Holt, Rinehart, Winston, New York.

Bleistein, N., 1984, Mathematical methods for wave phenomena: Academic Press, New York.

Bleistein, N., Cohen, J.K., and Hagin, F.G., 1984, Computational and asymptotic aspects of velocity inversion: Center for Wave Pheomena research report, CWP-004.

Bleistein, N., and Gray, S.H., 1984, An extension of the Born inversion method to a depth dependent reference profile: Center for Wave Phenomena research report CWP-007.

Carter, J.A. and Frazer, L.N., 1984, Accommodating lateral velocity changes in Kirchhoff migration by means of Fermat's principle: Geophysics, v.49, p.46-53.

Cohen, J.K., and Bleistein, N., 1979a, Velocity inversion procedure for acoustic waves: Geophysics, v.44, p.1077-1085.

_______, 1979b, The singular function of a surface and physical optics inverse scattering: Wave Motion, v.1, p.153-161.

_______, 1983, The influence of out-of-plane surface properties on unmigrated time sections: Geophysics, v.48, p.125-132.

Diebold, J.B. and Stoffa, P.L., 1981, The traveltime equation, tau-p mapping and inversion of common midpoint data: Geophysics, v.46, p.238-254.

Hagin, F.G. and Cohen, J.K., 1984, Refinements to the linear velocity inversion theory: Geophysics, v.49, p.112-118.

Keller, J.B., 1978, Rays, waves and asymptotics: Bull. Amer. Math. Soc., v.84, p.727-750.

APPENDIX A

NOTATIONS AND IDENTITIES

We define

$$n(z) = \frac{c(0)}{c(z)} \quad , \tag{A-1}$$

$$k_3(K,z) = \sqrt{n^2(z) - K^2} \quad , \tag{A-2}$$

and the integrals,

$$D(K,z) = \int_0^z k_3(K,\zeta)\,d\zeta \quad , \tag{A-3}$$

$$E(K,z) = \int_0^z \frac{d\zeta}{k_3(K,\zeta)} \quad , \tag{A-4}$$

$$F(K,z) = \int_0^z \frac{d\zeta}{k_3^3(K,\zeta)} \quad , \tag{A-5}$$

$$G(K,z) = \int_0^z \frac{n^2(\zeta)\,d\zeta}{k_3(K,\zeta)} \quad , \tag{A-6}$$

$$H(K,z) = \int_0^z \frac{n^2(\zeta)\,d\zeta}{k_z^2(K,\zeta)} \quad , \tag{A-7}$$

$$s(K,z) = \int_0^z \frac{n(\zeta)\,d\zeta}{k_z(K,\zeta)} \quad . \tag{A-8}$$

Similar quantities occur in the τ-p theory, see Diebold and Stoffa [1981]. Among the many relations which link these quantities, we cite below those that are useful in carrying out the calculations presented in this paper and its appendices.

First of all, from (A-2) it follows that

$$D + K^2E = G \quad , \tag{A-9}$$

$$E + K^2F = H \quad . \tag{A-10}$$

Next we cite the k and z partial derivatives of D, E and G which follow respectively from use of

$$\frac{\partial k_z}{\partial K} = -\frac{K}{k_z} \tag{A-11}$$

and the Fundamental Theorem of calculus:

$$D_K = -KE \quad , \quad D_z = k_z(K,z) \quad , \tag{A-12}$$

$$E_K = KF \quad , \quad E_z = \frac{1}{k_3} \tag{A-13}$$

$$G_K = KH \quad , \quad G_z = \frac{n^2}{k_3} \quad . \tag{A-14}$$

Finally, from (A-13) and (A-9) it follows that

$$(KE)_K = H \quad . \tag{A-15}$$

APPENDIX B

THE STRATIFIED MEDIA GREEN'S FUNCTION

Using Keller's [1978] "ray method", developed in the 1950's, we seek a high frequency approximation,

$$g(\underline{x},z) \sim A(\underline{x},z)\, e^{i\omega\tau(\underline{x},z)} \quad , \quad \underline{x} = (x,y) \tag{B-1}$$

which asymptotically satisfies the Helmholtz equation,

$$\nabla^2 g + \frac{\omega^2}{c^2(z)}\, g = -\delta(\underline{x}-\underline{\xi})\, \delta(z) \quad , \quad \underline{\xi} = (\xi,\eta) \quad . \tag{B-2}$$

To complete the specification of g, we insist that it behave like the free space (i.e. constant c) Green's function as the field point, $(\underline{x},z)$, approaches the source point, $(\underline{\xi},0)$. This entails the conditions,

$$\tau \rightarrow R/c(0) \quad , \quad A \rightarrow \frac{1}{4\pi R} \tag{B-3}$$

as $R \rightarrow 0$, where

$$R^2 = |\underline{x}-\underline{\xi}|^2 + z^2 \quad . \tag{B-4}$$

We substitute (B-1) into (B-2) and separately equate the coefficients of ω^2 and ω to zero (this is the high frequency approximation) giving rise to the <u>eikonal</u> equation,

$$\underline{k} \cdot \underline{k} = n^2(z) \quad , \quad \underline{k} = c(0)\, \nabla\tau \quad , \quad n = \frac{c(0)}{c(z)} \tag{B-5}$$

and the <u>transport</u> equation,

$$2\,\underline{k}\cdot\nabla A + (\nabla\cdot\underline{k})\,A = 0 \quad . \tag{B-6}$$

The former equation can be solved by the <u>method of characteristics</u> (see Bleistein, 1984) which reduces the problem to the solution of a system of ordinary differential equations. The first of these equations are:

$$\frac{d\underline{x}}{d\sigma} = \underline{K} \quad , \quad \frac{dz}{d\sigma} = k_3 \quad ; \quad \underline{K} \equiv (k_1,k_2) \quad , \ \underline{k} = (\underline{K},k_3) \tag{B-7}$$

$$\frac{d\underline{K}}{d\sigma} = 0 \quad , \quad \frac{dk_3}{d\sigma} = -\,n\,n'(z) \tag{B-8}$$

which define the <u>rays</u>, σ being the ray parameter. The source term in (B-2) makes

$$\underline{x}(0) = \underline{\xi} \quad , \quad z(0) = 0 \quad , \tag{B-9}$$

a natural choice as initial data for (B-7). The data for $\underline{k}(0)$ consists of an arbitrary unit vector (cf. (B-5), noting that $n(0) = 1$). To be specific we choose

$$\underline{K}(0)\ \text{arbitrary} \quad , \quad k_3(0) = \sqrt{1-K^2} \tag{B-10}$$

where we have introduced

$$K = |\underline{K}| = \sqrt{k_1^2 + k_2^2} \quad . \tag{B-11}$$

From (B-10) and the fact that we are not considering turned rays here, it follows that $\underline{K} = (k_1,\ k_2)$ can be viewed as the direction numbers of the rays initiating from $(\xi,\ \eta,\ 0)$.

Using (B-6) and (B-7) we find

$$\frac{d\tau}{d\sigma} = \frac{n^2}{c(0)} \tag{B-12}$$

$$2\,\frac{dA}{d\sigma} + (\nabla \cdot \underline{k})\,A = 0 \tag{B-13}$$

which together with the data (B-3) completes the specification of g, expressed in (B-1), in terms of a system of ordinary differential equations.

Proceeding to the analysis, we first note that by (B-8), $\underline{K} = \underline{K}(0)$, so henceforth we simply write $\underline{K}$ for this constant vector. Then the eikonal equation (B-5) gives us

$$k_3(K,z) = \sqrt{n^2 - K^2}\ . \tag{B-14}$$

Then (B-7) yields

$$\frac{d\underline{x}}{dz} = \frac{\underline{K}}{\sqrt{n^2 - K^2}} \tag{B-15}$$

or

$$\underline{x} - \underline{\xi} = \underline{K}\,E(K,z) \tag{B-16}$$

where E is defined by (A-3). Similarly, we find

$$\tau = \frac{1}{c(0)}\,G(K,z) \tag{B-17}$$

where G is defined by (A-6).

If we introduce the ray Jacobian,

$$J = \frac{\partial(\underline{x}, z)}{\partial(\underline{K}, \sigma)} \tag{B-18}$$

and use the fact that

$$\frac{dJ}{d\sigma} = J \nabla \cdot \underline{k} \quad , \tag{B-19}$$

then the transport equation (B-6) can be recast as

$$A \sqrt{J} = \text{constant} \quad . \tag{B-20}$$

Thus

$$A \sqrt{J} = \lim_{R \to 0} A \sqrt{J} \tag{B-21}$$

which by (B-3) implies that

$$A = \frac{1}{\sqrt{J}} \lim_{R \to 0} \left[\frac{\sqrt{J}}{4\pi R} \right] \quad . \tag{B-22}$$

We now indicate how to obtain the partial derivatives which are the elements of J.

Implicit differentiation of (B-17) with respect to k_1, k_2 and (A-14) yields

$$\frac{\partial z}{\partial k_j} = \frac{k_3 k_j H}{n^2} \quad , \quad j = 1,2 \quad . \tag{B-23}$$

Similarly from (B-16)

$$\frac{\partial x_i}{\partial k_j} = E\, \delta_{ij} + \left[F - \frac{H}{n^2} \right] k_i k_j \quad , \quad j = 1,2 \quad . \tag{B-24}$$

Since $\frac{\partial \underline{x}}{\partial \sigma}$ and $\frac{\partial z}{\partial \sigma}$ are given directly by (B-7), we are now able to form J.

A short calculation, involving the use of (A-10) yields

$$J = k_3 \; E \; H \quad . \tag{B-25}$$

It is easy to show that as $R \rightarrow 0$ (equivalently $\sigma \rightarrow 0$).

$$J \rightarrow \frac{z^2}{k_3^3} \quad . \tag{B-26}$$

In the same limit, (B-16) implies that

$$k_3 \rightarrow k_3(K,0) = \sqrt{1-K^2} = z/R \tag{B-27}$$

so that

$$J \rightarrow \frac{R^3}{z} = \frac{R^2}{k_3(K,0)} \quad . \tag{B-28}$$

Then (B-22) gives

$$A = \frac{1}{\sqrt{k_3(K,z)E(K,z)H(K,z)}} \cdot \frac{1}{4\pi\sqrt{k_3(K,0)}} \quad . \tag{B-29}$$

Since A and τ depend only on K, we need only employ the magnitude of (B-16) in the sequel:

$$\rho = KE(K,z) \quad ; \quad \rho \equiv |\underline{x} - \underline{\xi}| \quad . \tag{B-30}$$

Equation (B-14), (B-17), (B-29) and (B-30) are equivalent to equations (3-8) of the text.

Finally, we relate our parameter σ along the rays to the corresponding arclength parameter. From (B-7) we have

$$\frac{d\underline{x}}{d\sigma} \cdot \frac{d\underline{x}}{d\sigma} + \frac{dz}{d\sigma}\frac{dz}{d\sigma} = K^2 + k_3^2 = n^2 \tag{B-31}$$

and so

$$\frac{ds}{d\sigma} = n(z) \quad . \tag{B-32}$$

Using the second equation in (B-7) once more, we find

$$\frac{ds}{dz} = \frac{n}{k_3} \tag{B-33}$$

or

$$s = \int_0 \frac{n(\zeta)\,d\zeta}{k_3(K,\zeta)} \quad . \tag{B-34}$$

APPENDIX C

KIRCHHOFF DATA

The derivation presented in Cohen and Bleistein [1983] applies here with the constant c Green's function used there being replaced with the c(z) Green's function derived in Appendix B. Thus, equation (8) of that paper may be recast in our present notation as

$$u_S(\underline{z},\omega) \sim \int_s dS \; R \frac{\partial}{\partial n} g^2(\underline{x},z;\underline{\xi};\omega) \tag{C-1}$$

with g defined by equations (3-6) and the reflection coefficient R being defined by

$$R = \frac{\gamma - \gamma_1}{\gamma + \gamma_1} \tag{C-2}$$

where

$$\gamma = \hat{n} \cdot \nabla\tau \quad , \quad \gamma_1 = \mathrm{sgn}(\gamma) \sqrt{\gamma^2 + \frac{1}{c_1^2} - \frac{1}{c^2}} \quad , \tag{C-3}$$

and $\hat{n}$ is the upward normal to S. Here c_1 is defined by (35).

Since g has the form given in (B-1)

$$\frac{\partial}{\partial n} g^2 \sim 2i\omega \, \hat{n} \cdot \nabla\tau \; g^2 \quad , \tag{C-4}$$

subject to

$$\rho = K E(K, z) \quad . \tag{C-5}$$

Hence (B-1), with the definition of γ given in (C-3) and that of S given by (33), yields

$$\frac{\partial}{\partial n} g^2 \sim 2 i \omega \gamma S \; e^{2 i \omega \tau} \quad . \tag{C-6}$$

Thus (30) follows from the form, (29), of the surface. It remains to establish the detailed calculation of γ given in (32).

We have:

$$\begin{aligned}
\gamma &= \hat{n} \cdot \nabla\tau \\
&= \frac{1}{c(0)} \hat{n} \cdot \nabla G(K, z) \\
&= \frac{1}{c(0)} \frac{(h', 0, -1)}{\sqrt{1+(h')^2}} \cdot \left[G_K \frac{\partial K}{\partial x} \; , \; G_K \frac{\partial K}{\partial y}, \; G_K \frac{\partial K}{\partial z} + G_z \right] \\
&= \frac{1}{c(0)} \frac{1}{\sqrt{1+(h')^2}} \cdot \left[h' G_K \frac{\partial K}{\partial x} - G_K \frac{\partial K}{\partial z} - G_z \right]
\end{aligned} \tag{C-7}$$

From the constraint (C-5) we compute

$$\frac{\partial K}{\partial x} = \frac{x - \xi}{\rho (K E)_K} \quad , \quad \frac{\partial K}{\partial K} = - \frac{K E_z}{(K E)_K} \tag{C-8}$$

and then (32) follows from (C-7), (C-8) and the results of Appendix A.

APPENDIX D

STATIONARY PHASE CALCULATIONS

Assuming that the phase, $\Phi(\underline{x})$, has a single simple stationary point, $\underline{x}_s$:

$$\nabla\Phi(\underline{x}_s) = \underline{0} \quad , \quad \det \Phi_{x_i x_j}(\underline{x}_s) \neq 0 \tag{D-1}$$

the integral,

$$I(\lambda) \sim \int e^{i\lambda\Phi(\underline{x})} A(\underline{x}) d^n x \quad , \quad \lambda > 0 \tag{D-2}$$

can be evaluated asymptotically by the multidimensional stationary phase formula, (see Bleistein [1984] or Bleistein and Handelsman [1975]).

$$I(\lambda) \sim \left[\frac{2\pi}{\lambda} \right]^{n/2} \frac{A_s}{\sqrt{|\det\Phi_{ij}|}} \exp\{i\lambda\Phi_s + i \frac{\pi}{4} \operatorname{sig}\Phi_{ij}\} \quad , \tag{D-3}$$

Here A_s, Φ_s, and Φ_{ij}, denote respectively the amplitude A, the phase Φ and the Hessian matrix, $\Phi_{x_i x_j}$ evaluated at the stationary point, $\underline{x} = \underline{x}_s$. Further, sig Φ_{ij} denotes the <u>signature</u> of Φ_{ij}, i.e. the number of positive eigenvalues minus the number of negative eigenvalues.

If there were several stationary points, (D-5) would be replaced by a sum over the contributions from these several points. If the Hessian matrix vanished, (D-3) would have to be replaced by a more general result. However, only (D-3) is required here.

In our applications, instead of a positive parameter, λ, we have the signed quantity, $2\omega/c(0)$. The sign can be accounted for by replacing Φ

by $-\Phi$. In addition, only the case n = 4 occurs in the text, hence

$$I(\omega) \sim \left[\frac{\pi c(0)}{|\omega|} \right]^2 \frac{A_s}{\sqrt{|\det \Phi_{ij}|}} \exp\{2i \frac{\omega}{c(0)} \Phi_s + \frac{i\pi}{4} \operatorname{sgn}(\omega) \operatorname{sig} \Phi_{ij}\} \quad . \qquad \text{(D-4)}$$

The context for our asymptotic evaluations is

$$J = \int I(\omega)\, \omega^2 \, d\omega \qquad \text{(D-5)}$$

(more precisely a bandlimited version of (D-5)).

The most important case for us is when $\operatorname{sig}\Phi_{ij} = 0$, so that

$$J \sim \left[\pi c(0) \right]^2 \frac{A_s}{\sqrt{|\det \Phi_{ij}|}} \; \delta \left[\Phi_s \right] \; , \quad (\operatorname{sig}\Phi_{ij} = 0) \; , \qquad \text{(D-6)}$$

where δ denotes the Dirac delta function. The only other case that occurs below is $\operatorname{sig}\Phi_{ij}, = \pm 2$, in which case,

$$I(\omega) \sim \left[\frac{\pi c(0)}{|\omega|} \right]^2 \cdot (\pm\, i \operatorname{sgn}(\omega))\, e^{2 \frac{i\omega}{c(0)} \Phi_s} \; , \quad (\operatorname{sig}\Phi_{ij} = \pm 2) \; , \qquad \text{(D-7)}$$

and

$$J \sim \pi^2 c^2(0) \frac{A_s}{\sqrt{|\det \Phi_{ij}|}} \frac{1}{\Phi_s} \; , \quad (\operatorname{sig}\Phi_{ij} = \pm 2) \; . \qquad \text{(D-8)}$$

Although the evaluation of the signature of a symbolic 4 x 4 matrix can be tedious or impossible, our task is considerably simplified by the fact that our Hessian matrices have the special form,

$$\Phi = \begin{bmatrix} \alpha & 0 & v & 0 \\ 0 & \beta & 0 & u \\ v & 0 & \gamma & 0 \\ 0 & u & 0 & \delta \end{bmatrix} , \tag{D-9}$$

whence

$$\det \Phi = (\alpha\gamma - v^2) \quad (\beta\delta - u^2) \quad . \tag{D-10}$$

To evaluate the signature, we need to determine the roots, λ, of the eigenvalue equation,

$$\det (\Phi - \lambda I) = 0 \quad , \tag{D-11}$$

where I is the identity matrix. From (D-10), we find at once:

$$\det (\Phi - \lambda I) = \left[\lambda^2 - (\alpha+\gamma) \lambda + \alpha\gamma - v^2 \right] \left[\lambda^2 - (\beta+\delta) + \beta\delta - u^2 \right] \quad . \tag{D-12}$$

Thus,

$$v^2 > \alpha\gamma \quad , \quad u^2 > \beta\delta \quad \Rightarrow \quad \mathrm{sig}\Phi_{ij} = 0 \quad . \tag{D-13}$$

and similarly,

$$v^2 > \alpha\gamma \quad , \quad u^2 < \beta\delta \quad \Rightarrow \quad \mathrm{sig}\Phi_{ij} = \pm 2 \quad , \tag{D-14}$$

etc.

We now turn to the specific stationary phase calculations in the text. We shall freely use the results of Appendix A without explicit citation. First we examine the phase in (14):

$$\Phi(\underline{x},\underline{\xi}) = G(K,z) - G(K',z') \tag{D-15}$$

where $K(\rho,z)$, $K'(\rho',z')$ are defined by

$$\rho = KE(K,z) \quad , \quad \rho' = K'E(K',z') \tag{D-16}$$

with

$$\rho = |\underline{x} - \underline{\xi}| \quad , \quad \rho' = |\underline{x}' - \underline{\xi}| \quad . \tag{D-17}$$

Implicit differentiation of (D-16) with respect to x_i yields

$$\frac{x_i - \xi_i}{\rho} = (KE)_K \frac{\partial K}{\partial x_i} = H \frac{\partial K}{\partial x_i} \tag{D-18}$$

so

$$\frac{\partial K}{\partial x_i} = \frac{x_i - \xi_i}{KE(K,z)\,H(K,z)} \quad . \tag{D-19}$$

Similarly,

$$\frac{\partial K}{\partial \xi_i} = \frac{\xi_i - x_i}{KE(K,z)H(K,z)} \quad , \quad \frac{\partial K'}{\partial \xi_i} = \frac{\xi_i - x_i'}{K'E(K',z')H(K',z')} \quad . \tag{D-20}$$

Now using these results, we find that

$$\Phi_{x_i} = G_K \frac{\partial K}{\partial x_i} = \frac{x_i - \xi_i}{E(K,z)} \tag{D-21}$$

and similarly

$$\Phi_{\xi_i} = \frac{\xi_i - x_i}{E(K,z)} - \frac{\xi_i - x_j'}{E(K',z')} \quad . \tag{D-22}$$

From (D-21) and (D-22), we see that the stationarity conditions are:

$$\underline{x} = \underline{\xi} = \underline{x}' \quad . \tag{D-23}$$

Hence by (D-15),

$$\rho = \rho' = 0 \quad . \tag{D-24}$$

Now it is clear that

$$K = K' = 0 \tag{D-25}$$

is a solution of the constraint conditions (D-16). Moreover, since

$$(KE)_K = H > 0 \quad , \tag{D-26}$$

this is the <u>only</u> solution. We obtain for the phase at the stationary point:

$$\Phi_s = G(0,z) - G(0,z') = \int_{z'}^{z} n(\zeta)\, d\zeta \tag{D-27}$$

and for the components of the Hessian matrix:

$$\Phi_{ij} = \begin{bmatrix} \alpha & 0 & -\alpha & 0 \\ 0 & \alpha & 0 & -\alpha \\ -\alpha & 0 & \alpha-\alpha' & 0 \\ 0 & -\alpha & 0 & \alpha-\alpha' \end{bmatrix} \tag{D-28}$$

where

$$\alpha = 1/E(0,z) \quad , \quad \alpha' = 1/E(0,z') \quad . \tag{D-29}$$

In the notation of (D-7) we have

$$v^2 - \alpha\gamma = u^2 - \beta\delta = \alpha\alpha' > 0 \tag{D-30}$$

so that

$$\det \Phi_{ij} = \left[\frac{1}{E(0,z)\ E(0,z')} \right]^2 \quad , \quad \mathrm{sig}\Phi_{ij} = 0 \tag{D-31}$$

These results allow us to use (D-4) in (13) to obtain (21) and (D-6) to obtain (22).

We now turn to the stationary phase evaluation of (36). Here the phase is

$$\Phi = G(\bar{K},\bar{z}) - G(K,z) \quad , \tag{D-32}$$

subject to the usual constraints

$$\rho = |\underline{x} - \underline{\xi}| = K E(K,z) \quad , \quad \bar{\rho} = |\bar{\underline{x}} - \underline{\xi}| = \bar{K} E(\bar{K},\bar{z}) \quad . \tag{D-33}$$

This stationary phase evaluation is somewhat more difficult because $\bar{z}$ depends on $\bar{x}$:

$$\bar{z} = h(\bar{x}) \quad , \tag{D-34}$$

and hence the $\Phi_{\bar{x}}$ condition is more complicated than that in (D-21). The analogue of (D-18) is

$$\frac{\bar{x}-\xi}{\bar{\rho}} = (\bar{K}\bar{E})_K \frac{\partial\bar{K}}{\partial\bar{x}} + \bar{K}\bar{E}_z = \bar{H}\,\frac{\partial\bar{K}}{\partial\bar{x}} + \frac{\bar{K}}{\bar{k}_s}\,h' \quad . \tag{D-35}$$

so

$$\frac{\partial\bar{K}}{\partial\bar{x}} = \frac{\bar{x}-\xi}{\bar{K}\bar{E}\bar{H}} - \frac{\bar{K}h'}{\bar{k}_s H} \tag{D-36}$$

where we use notations like

$$\bar{E} = E(\bar{K},\bar{z}) \tag{D-37}$$

for quantities which depend on $\bar{K}$ and $\bar{z}$. The remaining partials of $\bar{K}$ and K have the same form as in (D-18) and (D-19) since $\bar{z}$ does not depend on $\bar{y}$, ξ, or y, and since z is a completely independent variable:

$$\frac{\partial\bar{K}}{\partial\bar{y}} = \frac{\bar{y}-\eta}{\bar{K}\bar{E}\bar{H}} \quad , \quad \frac{\partial\bar{K}}{\partial\xi} = \frac{\xi-\bar{x}}{\bar{K}\bar{E}\bar{H}} \quad , \quad \ldots \tag{D-38}$$

These results allow one to compute:

$$\begin{aligned} \Phi_{\bar{x}} &= \bar{G}_K \frac{\partial\bar{K}}{\partial\bar{x}} + \bar{G}_z\; h'(\bar{x}) \\ &= \frac{\bar{x}-\xi}{\bar{E}} - \frac{\bar{K}^2}{\bar{k}_s}\,h' + \frac{n^2(\bar{z})}{\bar{k}_s} \\ &= \frac{\bar{x}-\xi}{\bar{E}} + \bar{k}_s\; h' \quad , \end{aligned} \tag{D-39}$$

$$\Phi_{\bar{y}} = \frac{\bar{y}-\eta}{\bar{E}} \quad , \quad \Phi_{\xi} = \frac{\xi-\bar{x}}{\bar{E}} - \frac{\xi-x}{E} \quad , \quad \Phi_{\eta} = \frac{\eta-\bar{y}}{\bar{E}} - \frac{\eta-y}{E} \quad .$$

The $\bar{y}$, η derivatives give rise to the single stationarity conditions:

$$\bar{y} = \eta = y \tag{D-40}$$

Hence the constraints (D-30) reduce to

$$\rho = \left|x-\xi\right| = K\,E(K,z) \quad , \qquad \rho = \left|\bar{x}-\xi\right| = \bar{K}\,E(\bar{K},\bar{z}) \quad , \tag{D-41}$$

which can be rewritten as

$$x-\xi = \mu K E \quad , \qquad \mu \equiv \mathrm{sgn}(x-\xi) \tag{D-42}$$

and

$$\bar{x}-\xi = \bar{\mu}\bar{K}\bar{E} \quad , \qquad \bar{\mu} \equiv \mathrm{sgn}(\bar{x}-\xi) \quad . \tag{D-43}$$

These results allow us to simplify the remaining partials in (D-39) to

$$\bar{\Psi}_{\bar{x}} = \bar{\mu}\bar{K} + \bar{k}_s h' \tag{D-44}$$

and

$$\bar{\Phi}_{\bar{y}} = -\,\bar{\mu}\bar{K} + \mu K \quad . \tag{D-45}$$

For stationarity, we must have

$$\mu = \bar{\mu} = -\,\mathrm{sgn}(h') \tag{D-46}$$

and then also

$$K = \bar{K} = \bar{k}_s h' \quad . \tag{D-47}$$

The latter equality allows to determine that

$$\bar{K} = \frac{n(\bar{z}) \; |h'|}{\sqrt{1+h'^2}} \quad , \quad \bar{k}_s = \frac{n(\bar{z})}{\sqrt{1+h'^2}} \quad . \tag{D-48}$$

Finally (D-46) allows us to restate (D-42), (D-43) as

$$x - \xi = - \; \bar{K}\, E(\bar{K}, z) \; \mathrm{sgn}(h') \tag{D-49}$$

and

$$\underline{x} - \xi = - \bar{K}\, E(\bar{K}, \bar{z}) \; \mathrm{sgn}(h') \quad . \tag{D-50}$$

At the stationary point, the phase is

$$\Phi_s = G(\bar{K}, \bar{z}) - G(\bar{K}, z) \quad . \tag{D-51}$$

Further, a number of terms of the Hessian matrix vanish because of (D-40) leaving us with

$$\Phi_{ij} = \left. \begin{bmatrix} \Phi_{\bar{x}\bar{x}} & 0 & \Phi_{\bar{x}\xi} & 0 \\ 0 & \Phi_{\bar{y}\bar{y}} & 0 & \Phi_{\bar{y}\eta} \\ \Phi_{\bar{x}\xi} & 0 & \Phi_{\xi\xi} & 0 \\ 0 & \Phi_{\bar{y}\eta} & 0 & \Phi_{\eta\eta} \end{bmatrix} \right|_s \tag{D-52}$$

which has the form (D-9) with

$$v^2 - \alpha\gamma = (\Phi^2_{\bar{x}\xi} - \Phi_{\bar{x}\bar{x}}\, \Phi_{\xi\xi})\big|_s \quad , \tag{D-53}$$

and

$$u^2 - \beta\gamma = (\Phi^2_{\bar{y}\eta} - \Phi_{\bar{y}\bar{y}}\, \Phi_{\eta\eta})\big|_s \quad . \tag{D-54}$$

We observe that (D-36) and the stationarity conditions yield

$$\frac{\partial \bar{K}}{\partial \bar{x}}\bigg|_S = \frac{\bar{\mu}}{\bar{H}} - \frac{h'|h'|}{\bar{H}} = \frac{\bar{\mu}}{\bar{H}}(1+h'^2) \quad , \quad \bar{\mu} = -\mathrm{sgn}(h') \tag{D-55}$$

and similarly

$$\frac{\partial \bar{K}}{\partial \xi}\bigg|_S = -\frac{\bar{\mu}}{\bar{H}} \quad , \quad \frac{\partial K}{\partial \xi}\bigg|_S = \frac{-\bar{\mu}}{H(\bar{K},z)} \quad . \tag{D-56}$$

These facts allow to evaluate (D-53) as

$$v^2 - \alpha\gamma = \frac{(1+h'^2)^2}{H(\bar{K},z)H(\bar{K},z)} + \frac{n(\bar{z})h''}{\sqrt{1+h'^2}}\left[\frac{1}{H(\bar{K},z)} - \frac{1}{H(\bar{K},\bar{z})}\right] \tag{D-57}$$

and (D-54) as

$$u^2 - \beta\gamma = \frac{1}{E(\bar{K},z)E(\bar{K},\bar{z})} > 0 \quad . \tag{D-58}$$

Thus, "near" the reflector, $z = \bar{z}$, both $v^2 > \alpha\gamma$, $u^2 > \beta\delta$ hold and we have $\mathrm{sig}\bar{\Phi}_{ij} = 0$. In this regime, (D-6) applies and we see that we have a delta function "acting" at the reflector. On the other hand, far from the reflector, we may have $v^2 < \alpha\gamma$ leading to the distribution, (D-8). However, since z is "far" from $\bar{z}$, this distribution is regular, indeed "small", so that the results given in the text follow.

unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER CWP-008	2. GOVT ACCESSION NO. AD-A149222	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Born Inversion with a Stratified Reference Velocity		5. TYPE OF REPORT & PERIOD COVERED Technical
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Jack K. Cohen and Frank G. Hagin		8. CONTRACT OR GRANT NUMBER(s) N00014-84-K-0049
9. PERFORMING ORGANIZATION NAME AND ADDRESS Center for Wave Phenomena Department of Mathematics Colorado School of Mines, Golden, CO 80401		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR SRO-159/84APR20 (411)
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Arlington, VA 22217		12. REPORT DATE December 8, 1984
		13. NUMBER OF PAGES 42
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report)
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

This document has been approved for public release and sale; its distribution is unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Seismic Inversion, Born, Stratified Medium, High Frequency, Asymptotics.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

see reverse side

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-014-6601 |

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

ABSTRACT

Carter and Frazer [1984], and Bleistein and Gray [1984], presented inversion algorithms which included the effect of a stratified reference velocity, c(z). Those papers did not address the question of obtaining accurate values of the reflection coefficient; this is the issue treated here. Thus, in the language of Bleistein, Cohen and Hagin [1984], the earlier algorithms provided structural inversions, the location of the subsurface layers; whereas the present algorithm also provides an accurate estimate of the reflectivity function, which depicts the reflectors and provides an estimate of the reflection strengths across the layers.

Since we employ a perturbation assumption (the "Born Approximation"), the constant reference speed inversion first described in Bleistein and Cohen [1979a] and reviewed in BCH, is often not adequate at depth. Although recursive use of the algorithm is possible and although the results can be significantly enhanced by suitable pre- or post-processing (e.g., see Hagin and Cohen [1984]), extension of the perturbation method to a stratified reference profile is highly significant. It is far more likely that the actual velocity function can be well approximated by a stratified reference velocity than by a constant one, which in turn enhances the validity of the perturbation assumption and the inversion results.

END

FILMED

2-85

DTIC